The Multigroup Common Factor Model With Minimal Uniqueness Constraints and the Power to Detect Uniform Bias

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An alternative formulation of the multigroup common factor model with minimal uniqueness constraints is considered. This alternative formulation is based on a simple identification constraint that is related to the standard maximum likelihood constraint used in single-group common factor analysis. It is argued that the alternative formulation leads to less technical difficulties in applications than earlier formulations of this multigroup common factor model. Furthermore, associated tests for various measurement invariance constraints across groups are proposed, such as an omnibus test for the absence of uniform bias. By means of an empirical example, the fitting of several multigroup common factor models with minimal uniqueness constraints and the testing for measurement invariance over groups are demonstrated. The nesting of multigroup confirmatory factor models under the multigroup common factor model with minimal uniqueness constraints is also discussed. Finally, a small study is performed to investigate the drop in power to detect uniform bias in using the multigroup common factor model with minimal uniqueness constrains instead of a confirmatory special case. The results of the study show a small drop in power under all research conditions. *Index terms: multigroup common factor analysis, exploratory factor analysis, confirmatory factor model, measurement invariance, uniform bias, power*

Multigroup confirmatory factor modeling can be used to investigate hypothesized group differences in psychological variables, such as cognitive abilities and personality traits (Jöreskog, 1971; Sörbom, 1974). In addition, multigroup confirmatory factor modeling can be used to investigate the observed indicator variables that are used to measure the unobservable psychological variables. In this article, the interest is in factor modeling techniques that can be applied when the observed indicator variables are continuously distributed, such as in case of continuous item scores or continuous (sub)test scores. When in practice, the requirement of continuity is met, or when it is reasonable to treat each indicator variable as a continuous variable, then this multigroup confirmatory factor modeling approach provides a means to address two related questions: (a) Are the observed variables biased with respect to group membership? If they are unbiased, (b) how exactly do the groups differ with respect to the common factors that the observed variables measure (e.g., Dolan & Molenaar, 1994)? Here, the term *bias* is used for differential test functioning (DTF) when the indicator variables are (sub)test scores and for differential item functioning (DIF) when the indicator variables are continuous item scores (Shealy & Stout, 1993). In both cases, bias means that at least one of the indicator variables does not measure the same psychological variable(s) in the same way in different subgroups.

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Multigroup confirmatory factor modeling is well suited to address the two questions mentioned in the previous paragraph. First, it is theoretically well developed (Meredith, 1993; Mellenbergh, 1994a, 1994b). Mellenbergh (1994b) presented the common factor model as a latent trait model for continuously distributed item scores. This interpretation identifies the common factor model as a special case of a general item response model (Mellenbergh, 1994a). Mellenbergh defined, within the common factor model, central terms such as specific objectivity, item and test information, and item bias. Meredith (1993) derived the necessary constraints in the multigroup confirmatory factor model for the observed scores to be strict factorially invariant over groups. Strict factorial invariance is a necessary condition for unbiasedness, as defined by Mellenbergh. Second, multigroup confirmatory factor modeling is statistically well developed in the sense that (maximum likelihood) estimation and testing do not pose a problem, nesting among increasingly complex models is well understood (Horn & McArdle, 1992; Widaman & Reise, 1997), and suitable software is available and well disseminated—for example, the LISREL (Jöreskog & Sörbom, 2002) and Mx (Neale, Boker, Xie, & Maes, 1999) computer programs.

In almost all multigroup applications of common factor models, a confirmatory model is used. Usually, the specified model has sufficient substantively motivated zero factor loadings to ensure identification (i.e., to avoid rotational indeterminacy). Often, the confirmatory model also includes configural invariance over groups, which means that for each group, the same factor loadings are assumed to be zero. In addition, a number of increasingly restrictive models can be fitted in successive steps to test for the invariance over groups of (a) factor loadings, (b) residual variance components, and (c) indicator intercepts. The most restrictive multigroup confirmatory factor model (MG-CFM), under which the indicators may be viewed as unbiased with respect to group, is the strict factorial invariance model (Meredith, 1993). Under the strict factorial invariance model, all parameters are assumed to be invariant over groups except for the common factor covariances and factor means. This means that the observed group differences are a function of common factor differences.

Where at the start of a simultaneous factor analysis in several groups a well-developed confirmatory factor model is absent, an exploratory factor analysis can be carried out to obtain a satisfactory factor structure. This usually involves the testing of a sequence of common factor models with different numbers of factors in each group separately. Then, after selection of the number of common factors, orthogonal and/or oblique rotations are needed to arrive at an interpretable factor structure, in which certain factor loadings are so close to zero that they can be fixed to zero. Next, the finally selected factor structure can be used to carry out tests of constraints associated with strict factorial invariance. A well-known disadvantage of such an exploratory factor analysis is the possibility of capitalization on chance (i.e., peculiarities in the data of the particular sample may influence the selection of a final factor model). So if in the case of multiple groups the multigroup common factor model is selected on the basis of an exploratory factor analysis, then subsequent tests of constraints associated with strict factorial invariance might be affected by model misspecifications that resulted from capitalization on chance.

McArdle and Cattell (1994) proposed multigroup common factor models in which minimal constraints needed for identification are defined. These models can be viewed as basic multigroup common factor models with only restrictions on the number of common factors and additional constraints for the elimination of rotational indeterminacy. These models can be very useful in multigroup studies because they provide a means to investigate the various degrees of factorial invariance without imposing a confirmatory factor structure. An advantage of adopting such a model for this purpose is that the adverse effects of misspecified zero factor loadings are avoided.

In applications, however, the use of the specific formulation proposed by McArdle and Cattell (1994) can yield computational difficulties, as discussed by McArdle and Cattell and illustrated by

Millsap (2001) in the single-group case. As shown in the following section, the formulation of McArdle and Cattell is not based on certain optimal conditions. In this article, another formulation of the basic multigroup common factor model with minimal uniqueness constraints is considered. This alternative formulation is based on a simple identification constraint that is related to the standard maximum likelihood (ML) constraint used in single-group common factor analysis. It is argued that this alternative formulation of the multigroup common factor model with minimal uniqueness constraints has a significant advantage in applications over the one proposed by McArdle and Cattell.

The outline of this article is as follows. Both previously mentioned model formulations of the multigroup common factor model with minimal constraints are discussed in the next section. Then, various tests for the investigation of measurement invariance under this basic model are proposed. The first test, which will be discussed briefly, can serve as a test for factorial congruence, thus offering an alternative for the usual congruence measure (Chan, Ho, Leung, Chan, & Yung, 1999). The focus in this article is, however, more on the second test. This second test can be viewed as an omnibus test for the absence of uniform bias. Next, the fitting of the multigroup common factor model with minimal uniqueness constraints and the application of the various tests for measurement invariance are demonstrated by means of an empirical example. Subsequently, it is shown that any restrictive MG-CFM is nested under the multigroup common factor model with minimal uniqueness constraints and the same number of hypothesized common factors. Finally, the drop in statistical power is investigated to detect uniform bias in using the multigroup common factor model with minimal uniqueness constraints instead of the true MG-CFM. The article concludes with a brief discussion.

The Multigroup Common Factor Model With Minimal Uniqueness Constraints

Let $\mathbf{y}_{ig}$ denote the p-dimensional vector of observed scores of the randomly selected subject i from group g. The multigroup common factor model for the vector of observed scores is given by

$$\mathbf{y}_{ig} = \boldsymbol{\nu}_g + \boldsymbol{\Lambda}_g \boldsymbol{\eta}_{ig} + \boldsymbol{\varepsilon}_{ig}, \quad \text{for } g = 1, \ldots, G, \tag{1}$$

where $\boldsymbol{\nu}_g$ is a $(p \times 1)$ vector of constant intercept terms in group g, $\boldsymbol{\Lambda}_g$ is a $(p \times q)$ matrix of constant factor loadings in group g, $\boldsymbol{\eta}_{ig}$ is the q-dimensional random vector of common factor scores $(q < p)$ of subject i in group g, and $\boldsymbol{\varepsilon}_{ig}$ is the $(p \times 1)$ random vector of specific scores unique to each observed score of subject i in group g. It is assumed that $\boldsymbol{\eta}_{ig} \sim N_q(\boldsymbol{\alpha}_g, \boldsymbol{\Psi}_g)$ and $\boldsymbol{\varepsilon}_{ig} \sim N_p(\mathbf{0}, \boldsymbol{\Theta}_g)$, in each group, which implies that $\mathbf{y}_{ig}$ has also a multivariate normal distribution in each group. Because the vector $\boldsymbol{\mu}_g$ of observed means is defined as $E[\mathbf{y}_{ig}]$, it follows that

$$\boldsymbol{\mu}_g = \boldsymbol{\nu}_g + \boldsymbol{\Lambda}_g \boldsymbol{\alpha}_g, \quad \text{for } g = 1, \ldots, G. \tag{2}$$

The covariances between the common factor scores and the specific scores are assumed to equal zero, that is, $E[(\boldsymbol{\eta}_g - \boldsymbol{\alpha}_g)\boldsymbol{\varepsilon}_{ig}^t] = \mathbf{0}$. Then, because each $(p \times p)$ subpopulation covariance matrix $\boldsymbol{\Sigma}_g$ of the observed scores is defined as $E[(\mathbf{y}_{ig} - \boldsymbol{\mu}_g)(\mathbf{y}_{ig} - \boldsymbol{\mu}_g)^t]$, it follows that

$$\boldsymbol{\Sigma}_g = \boldsymbol{\Lambda}_g \boldsymbol{\Psi}_g \boldsymbol{\Lambda}_g^t + \boldsymbol{\Theta}_g, \quad \text{for } g = 1, \ldots, G, \tag{3}$$

where $\boldsymbol{\Psi}_g$ is the $(q \times q)$ subpopulation covariance matrix of the common factor scores, which equals $E[(\boldsymbol{\eta}_g - \boldsymbol{\alpha}_g)(\boldsymbol{\eta}_g - \boldsymbol{\alpha}_g)^t]$. The specific scores are assumed to be uncorrelated. Therefore, $\boldsymbol{\Theta}_g$ is a $(p \times p)$ diagonal matrix with the variances of the specific scores on the diagonal.

Without any additional constraints, the parameters of the model defined by equations (1), (2), and (3) are not identified. As always, both the origins and measurement scale units of the common factors must be determined. Because also $\boldsymbol{\Sigma}_g = \boldsymbol{\Lambda}_g \boldsymbol{\Psi}_g \boldsymbol{\Lambda}_g^t + \boldsymbol{\Theta}_g = \boldsymbol{\Lambda}_g^n \boldsymbol{\Psi}_g^n (\boldsymbol{\Lambda}_g^n)^t + \boldsymbol{\Theta}_g$, where $\boldsymbol{\Lambda}_g^n = \boldsymbol{\Lambda}_g \mathbf{M}_g$, $\boldsymbol{\Psi}_g^n = \mathbf{M}_g^{-1} \boldsymbol{\Psi}_g (\mathbf{M}_g^t)^{-1}$, for all g, and $\mathbf{M}_g$ is any $(q \times q)$ nonsingular transformation matrix, for all g, for each group q^2 restrictions must be imposed on the model to eliminate rotational indeterminacy. In what follows, two different ways of defining minimal constraints needed for identification are discussed. These two ways lead to different formulations of a mathematically equivalent model. That is, each way of defining minimal constraints yields another solution, but each solution can be obtained from the other solution by rotation, provided that both solutions are proper. Therefore, the values of the fit measures, the values of the fit indices, and the degrees of freedom will be the same for both solutions. In addition, subsequent rotation to enhance interpretability, such as varimax or promax rotation to approximate simple structure with one and only one nonzero element per row of the factor pattern matrix $\boldsymbol{\Lambda}_g$, will yield identical results regardless of how the minimal constraints are defined.

In both model formulations, the origins of the common factors are determined by setting $\boldsymbol{\alpha}_g = \mathbf{0}$, for all g. The first model formulation is now obtained by also applying a reference variable rotation (McArdle & Cattell, 1994) in each group. If $\boldsymbol{\Lambda}_g = \begin{bmatrix} \boldsymbol{\Lambda}_g^s \\ \boldsymbol{\Lambda}_g^u \end{bmatrix}$, for all g, and the transformation matrix $\mathbf{M}_g$ is selected to be equal to $(\boldsymbol{\Lambda}_g^s)^{-1}$, then $\boldsymbol{\Lambda}_g^n = \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\Lambda}_g^* \end{bmatrix}$, for all g, where $\mathbf{I}$ is the identity matrix with 1s on the main diagonal and 0s elsewhere, and $\boldsymbol{\Lambda}_g^* = \boldsymbol{\Lambda}_g^u (\boldsymbol{\Lambda}_g^s)^{-1}$, for all g. It can be shown that if $\mathbf{M}_g = (\boldsymbol{\Lambda}_g^s)^{-1}$, for all g, then $|(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n| > 0$, for all g. By means of this reference variable rotation, the factors are automatically scaled by the unities on the main diagonal of $\mathbf{I}$. So the first formulation of the multigroup common factor model with minimal constraints is

$$\left\{ \begin{array}{l} \boldsymbol{\mu}_g = \boldsymbol{\nu}_g, \\ \boldsymbol{\Sigma}_g = \boldsymbol{\Lambda}_g^n \boldsymbol{\Psi}_g^n (\boldsymbol{\Lambda}_g^n)^t + \boldsymbol{\Theta}_g, \end{array} \right\} \text{ and } \boldsymbol{\Lambda}_g^n = \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\Lambda}_g^* \end{bmatrix}, \text{ for } g = 1, \ldots, G. \tag{4}$$

This is an oblique case in which the elements of $\boldsymbol{\Psi}_g^n$ are unconstrained and q^2 elements are specified in $\boldsymbol{\Lambda}_g^n$, for all g.

McArdle and Cattell (1994) discuss other ways of imposing q^2 restrictions on the model than by means of a reference variable rotation. Because, in contrast to a reference variable rotation, these other possibilities do not guarantee that $|(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n| > 0$, for all g, they are not discussed here. In the case of a reference variable rotation, however, the locations of the q^2 specified elements in $\boldsymbol{\Lambda}_g^n$, for all g, are still not completely arbitrary. The rows of $\boldsymbol{\Lambda}_g$, for all g, can be arranged in $p \bullet (p-1) \cdot \ldots \cdot 1 = p!$ possible ways. So there are $\frac{p!}{(p-q)!} = p \bullet (p-1) \cdot \ldots \cdot (p-q+1)$ possible ways of selecting the $(q \times q)$ matrix $\boldsymbol{\Lambda}_g^s$, for all g, that all yield $|(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n| > 0$, for all g. For example, in the case of 13 observed indicator variables and 3 common factors, this means $13 \cdot \ldots \cdot (13-3+1) = 13 \bullet 12 \bullet 11 = 1{,}716$ possible ways. Each of these possible selections of $\boldsymbol{\Lambda}_g^s$ guarantees that $|(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n| > 0$, for all g, but the selection that maximizes $|(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n|$, for all g, is not necessarily obtained by one of these possible reference variable rotations. This is important to note because when the determinant of the matrix $(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n$ reaches a maximum value, for all g, the residual covariance matrix of each group is Gramian, which means in practice that corresponding computational procedures do not yield Heywood cases, that is, negative values of the diagonal elements of $\boldsymbol{\Theta}_g$, for all g (Browne, 1969). So selecting $\boldsymbol{\Lambda}_g^s$, for all g, such that $|(\boldsymbol{\Lambda}_g^n)^t \boldsymbol{\Lambda}_g^n|$ reaches a maximum value, for all g, guarantees a proper solution, whereas a reference variable rotation can yield an improper solution (Millsap, 2001).

In the second model formulation, the scale units of the common factors are determined by setting the diagonal elements of $\boldsymbol{\Psi}_g$, for all g, equal to 1. Because it is also assumed that the factors are orthogonal, the matrix $\boldsymbol{\Psi}_g$ equals the identity matrix $\mathbf{I}$, for all g. Again, the parameters of the model are still not identified because postmultiplying the matrix of factor loadings by an arbitrary orthogonal $(q \times q)$ matrix $\mathbf{M}_g$ leaves the population covariance matrix of the observed scores invariant, that is, $\boldsymbol{\Sigma}_g = \boldsymbol{\Lambda}_g\boldsymbol{\Lambda}_g^t + \boldsymbol{\Theta}_g = \boldsymbol{\Lambda}_g\mathbf{M}_g(\boldsymbol{\Lambda}_g\mathbf{M}_g)^t + \boldsymbol{\Theta}_g = \boldsymbol{\Lambda}_g\mathbf{M}_g\mathbf{M}_g^t\boldsymbol{\Lambda}_g^t + \boldsymbol{\Theta}_g$, for all g, where $\mathbf{M}_g\mathbf{M}_g^t = \mathbf{I}$, for all g. Now, it will be shown that to accomplish identification, the matrix $\boldsymbol{\Lambda}_g^t\boldsymbol{\Lambda}_g$ can be restricted to be diagonal, for all g. Because the matrix $\boldsymbol{\Sigma}_g - \boldsymbol{\Theta}_g$, for all g, is symmetric, it may be expressed in the form $\boldsymbol{\Omega}_g\boldsymbol{\Delta}_g\boldsymbol{\Omega}_g^t$, where $\boldsymbol{\Delta}_g$ is a diagonal matrix of order q and where $\boldsymbol{\Omega}_g$ is a $p \times q$ matrix with orthonormal columns satisfying $\boldsymbol{\Omega}_g^t\boldsymbol{\Omega}_g = \mathbf{I}$. The elements of $\boldsymbol{\Delta}_g$ are the q positive and distinct eigenvalues of $\boldsymbol{\Sigma}_g - \boldsymbol{\Theta}_g$. Consequently, $\boldsymbol{\Omega}_g$ is uniquely determined, for all g. Then, it follows that $\boldsymbol{\Lambda}_g$, for all g, is uniquely defined by $\boldsymbol{\Lambda}_g = \boldsymbol{\Omega}_g\boldsymbol{\Delta}_g^{\frac{1}{2}}$, for all g, because $\boldsymbol{\Lambda}_g\boldsymbol{\Lambda}_g^t = \boldsymbol{\Omega}_g\boldsymbol{\Delta}_g\boldsymbol{\Omega}_g^t = \boldsymbol{\Omega}_g\boldsymbol{\Delta}_g^{\frac{1}{2}}\boldsymbol{\Delta}_g^{\frac{1}{2}}\boldsymbol{\Omega}_g^t = \boldsymbol{\Omega}_g\boldsymbol{\Delta}_g^{\frac{1}{2}}(\boldsymbol{\Omega}_g\boldsymbol{\Delta}_g^{\frac{1}{2}})^t$. Finally, $\boldsymbol{\Lambda}^t\boldsymbol{\Lambda} = (\boldsymbol{\Omega}_g\boldsymbol{\Delta}_g^{\frac{1}{2}})^t\boldsymbol{\Omega}_g\boldsymbol{\Delta}_g^{\frac{1}{2}} = \boldsymbol{\Delta}_g^{\frac{1}{2}}\boldsymbol{\Omega}_g^t\boldsymbol{\Omega}_g\,\boldsymbol{\Delta}_g^{\frac{1}{2}} = \boldsymbol{\Delta}_g$, for all g. So to accomplish identification, the matrix $\boldsymbol{\Lambda}_g^t\boldsymbol{\Lambda}_g$ can be restricted to be diagonal, for all g. Note that from this constraint, it directly follows that the determinant $|\boldsymbol{\Lambda}_g^t\boldsymbol{\Lambda}_g|$ takes on its maximum value, for all g. A similar constraint is that the matrix $\boldsymbol{\Lambda}_g^t\boldsymbol{\Theta}_g^{-1}\boldsymbol{\Lambda}_g$ is restricted to be diagonal, for all g. This is the multigroup version of the standard ML constraint in single-group common factor analysis (Lawley & Maxwell, 1971). Presumably, McArdle and Cattell (1994) avoided this standard ML constraint for computational reasons. However, at present, the Mx program (Neale et al., 1999) can handle this constraint easily, as well as the constraint that $\boldsymbol{\Lambda}_g^t\boldsymbol{\Lambda}_g$ is diagonal, for all g.

Imposing again $\boldsymbol{\alpha}_g = \mathbf{0}$, for all g, on the model defined by equations (1) through (3), together with the identification constraints mentioned in the previous paragraph, yields

$$\boldsymbol{\mu}_g = \boldsymbol{\nu}_g, \quad \text{for } g = 1, \ldots, G, \tag{5}$$

and

$$\boldsymbol{\Sigma}_g = \boldsymbol{\Lambda}_g\boldsymbol{\Lambda}_g^t + \boldsymbol{\Theta}_g, \quad \text{for } g = 1, \ldots, G, \tag{6}$$

where both $\boldsymbol{\Theta}_g$ and $\boldsymbol{\Lambda}_g^t\boldsymbol{\Lambda}_g$ are diagonal, for all g. This second formulation of the multigroup common factor model with minimal uniqueness constraints, which will be designated Model 1 in the rest of this article, is preferred to the first model formulation and serves as a point of departure in what follows.

Tests for Factorial Invariance

For a meaningful comparison of multiple groups, additional invariance constraints across groups must be imposed on Model 1. In a multigroup common factor model, the constraint $\boldsymbol{\Lambda}_g = \boldsymbol{\Lambda}$, for all g, is necessary for metric or weak factorial invariance of the observed scores over groups (Widaman & Reise, 1997). The simultaneous constraints $\boldsymbol{\nu}_g = \boldsymbol{\nu}$, $\boldsymbol{\Lambda}_g = \boldsymbol{\Lambda}$, for all g, are necessary for the observed scores to be strong factorially invariant over groups (Meredith, 1993). For strict factorial invariance, the additional constraint $\boldsymbol{\Theta}_g = \boldsymbol{\Theta}$, for all g, is required (Meredith, 1993). These three factorial invariance constraints can be tested successively.

First, one can test whether $\boldsymbol{\Lambda}_g = \boldsymbol{\Lambda}$, for all g. This provides a model-based omnibus test for factorial congruence, which does not depend on a confirmatory factor structure. Factorial congruence is usually established by means of a congruence measure (Chan et al., 1999). This congruence measure is not usually subjected to any statistical test, it is unclear how large it should be to conclude that congruence holds to reasonable approximation, and it has to be calculated for each pair of common factors. Judging by the literature, any value between .9 and 1 is viewed as indicative of congruence. Only recently has the issue of statistical testing been considered by Chan et al. (1999), who proposed a bootstrap procedure. The test for factorial congruence considered here has a number of advantages over the use of the congruence measure. The test is based on a likelihood ratio test statistic, the test can be applied in situations with more than two groups, modification indices are available to investigate possible sources of misfit, and the problem of missing data (MAR and MCAR) can be handled well by raw data likelihood estimation (Schafer & Graham, 2002).

Now, because of the restriction $\mathbf{\Lambda}_g = \mathbf{\Lambda}$, for all g, the matrix $\mathbf{\Psi}_g$, for $g = 2, \ldots, G$, does not necessarily equal $\mathbf{I}$ for identification, and the elements of the symmetric matrix $\mathbf{\Psi}_g$, for $g = 2, \ldots, G$, can be estimated. The model with this first invariance restriction across groups is designated Model 2 and is written as

$$\boldsymbol{\mu}_g = \boldsymbol{\nu}_g, \quad \text{for } g = 1, \ldots, G, \tag{7}$$

and

$$\begin{cases} \mathbf{\Sigma}_1 = \mathbf{\Lambda}\mathbf{\Lambda}^t + \mathbf{\Theta}_1 \\ \mathbf{\Sigma}_g = \mathbf{\Lambda}\mathbf{\Psi}_g\mathbf{\Lambda}^t + \mathbf{\Theta}_g, \quad \text{for } g = 2, \ldots, G, \end{cases} \tag{8}$$

with the restriction that $\mathbf{\Lambda}^t\mathbf{\Lambda}$ is diagonal.

Second, one can test whether $\boldsymbol{\nu}_g = \boldsymbol{\nu}$, for all g, which provides an omnibus test for the absence of uniform bias independent of a confirmatory factor structure. By setting $\boldsymbol{\nu}_g = \boldsymbol{\nu}$, for all g, the vector $\boldsymbol{\alpha}_g$ of factor means, for Group $g = 2, \ldots, G$, can be estimated. Because the mean vector $\boldsymbol{\alpha}_1$, for arbitrarily selected Group 1, is kept equal to $\mathbf{0}$, the vector $\boldsymbol{\alpha}_g$, for group $g = 2, \ldots, G$, equals the vector of factor mean differences between Group 1 and Group $g = 2, \ldots, G$. So now the model for the vector of observed group means becomes

$$\begin{cases} \boldsymbol{\mu}_1 = \boldsymbol{\nu} \\ \boldsymbol{\mu}_g = \boldsymbol{\nu} + \mathbf{\Lambda}\boldsymbol{\alpha}_g, \quad \text{for } g = 2, \ldots, G, \end{cases} \tag{9}$$

and for the observed subpopulation covariance matrix, the model is

$$\begin{cases} \mathbf{\Sigma}_1 = \mathbf{\Lambda}\mathbf{\Lambda}^t + \mathbf{\Theta}_1 \\ \mathbf{\Sigma}_g = \mathbf{\Lambda}\mathbf{\Psi}_g\mathbf{\Lambda}^t + \mathbf{\Theta}_g, \quad \text{for g} = 2, \ldots, \text{G}, \end{cases} \tag{10}$$

with again the computationally convenient restriction that $\mathbf{\Lambda}^t\mathbf{\Lambda}$ is diagonal. This model is designated Model 3 and satisfies the conditions of strong factorial invariance.

Finally, it can be tested whether $\mathbf{\Theta}_g = \mathbf{\Theta}$, for all g, which is required for strict factorial invariance. This model is designated Model 4, and under this model, the vector of observed group means is

$$\begin{cases} \boldsymbol{\mu}_1 = \boldsymbol{\nu} \\ \boldsymbol{\mu}_g = \boldsymbol{\nu} + \mathbf{\Lambda}\boldsymbol{\alpha}_g, \quad \text{for g} = 2, \ldots, \text{G}, \end{cases} \tag{11}$$

and the observed group covariance matrix is

$$\begin{cases} \mathbf{\Sigma}_1 = \mathbf{\Lambda}\mathbf{\Lambda}^t + \mathbf{\Theta} \\ \mathbf{\Sigma}_g = \mathbf{\Lambda}\mathbf{\Psi}_g\mathbf{\Lambda}^t + \mathbf{\Theta}, \quad \text{for g} = 2, \ldots, \text{G}, \end{cases} \tag{12}$$

with again the restriction that $\mathbf{\Lambda}^t\mathbf{\Lambda}$ is diagonal.

Before discussing the nesting of multigroup common factor models and carrying out power calculations, the fitting of Model 1 through Model 4 to a real data set is illustrated in the next section.

Illustrative Example

In this illustrative example, the four multigroup common factor models discussed in the two preceding sections are fitted to a data set published by Jensen and Reynolds (1982). Previously, Dolan (2000) also used this data set to demonstrate the use of MG-CFM in investigating Spearman's hypothesis. The sample consists of 1,868 Whites and 305 Blacks. The data are the examinees' observed scores on the following 13 subscales of the Wechsler Intelligence Scale for Children–Revised (WISC-R): Information (I), Similarities (S), Arithmetic (A), Vocabulary (V), Comprehension (C), Digit Span (DS), Tapping Span (TS), Picture Completion (PC), Picture Arrangement (PA), Block Design (BD), Object Assembly (OA), Coding (CO), and Mazes (MA). For details of the sample, see Jensen and Reynolds.

First, a sequence of two-group common-factor models with minimal uniqueness constraints is fitted to the data to extract the number of common factors. Initially, a three-factor model with a verbal, a performance, and a memory span factor could have been selected on the basis of theoretical considerations (Jensen & Reynolds, 1982) and previous empirical research (Dolan, 2000). However, for completeness and illustrative purposes, two-group one-, two-, three-, and four-factor models with minimal uniqueness constraints are fitted to the data successively.

The analyses are carried out with the Mx computer program (Neale et al., 1999). The advantage of using this program over others, such as LISREL, is that it can easily handle the constraint that $\mathbf{\Lambda}^t\mathbf{\Lambda}$ is diagonal. The following standard fit indices of Mx are used: the chi-square value of the minimum fit function with its degrees of freedom, Akaike's information criterion (AIC), and the root mean square error of approximation (RMSEA). In addition, values of the Bayesian information criterion (BIC) are reported. Note that the degrees of freedom for a multigroup common factor model with minimal uniqueness constraints are obtained as follows. In all, there are $G \times (\frac{p(p+1)}{2} + p)$ observed statistics—that is, $\frac{p(p+1)}{2}$ observed covariances and p observed means—for each group. The total number of parameters to be estimated is $G \times ((p \times q) + 2p)$—that is, $(p \times q)$ factor loadings, p specific score variances, and p intercept terms—for each group. Furthermore, because $\mathbf{\Lambda}_g^t\mathbf{\Lambda}_g$ must be diagonal, for each group, there are $G \times (\frac{q(q-1)}{2})$ additional constraints that must be subtracted from the total number of parameters to be estimated. Consequently, there are $G \times (\frac{p(p-1)+q(q-1)}{2} - pq)$ degrees of freedom. The fitting results of the four common factor models (Model 1 with successively one, two, three, and four factors) are reported in Table 1.

From the results in Table 1, it can be concluded that the multigroup common factor Model 1 with either one or two factors does not fit the data. According to the values of the RMSEA, Model 1 with either three or four common factors gives an adequate description. Because the simpler model with three common factors has the smallest BIC value and because of reasons given previously, this model is selected for further analysis. In Table 2, the results are given for the three-factor Models 2, 3, and 4 with invariance constraints over the two groups.

The successive chi-square difference tests for $\mathbf{\Lambda}_g = \mathbf{\Lambda}$, $\boldsymbol{\nu}_g = \boldsymbol{\nu}$, and $\mathbf{\Theta}_g = \mathbf{\Theta}$, for all g, could be carried out, but chi-square values are inflated by large total sample sizes. Therefore, in the case of the present sample sizes, chi-square difference results are of little use. However, according to the values of the RMSEA, which is less dependent on the sample size, all four models fit the data well because all RMSEA values are below the value of 0.05 for a good fit. Then, because the values of the AIC and the BIC of Model 4 are the smallest, it can be concluded that strict factorial invariance can be retained (Mx code for Model 4 is given in the appendix).

Table 1
Values of the Fit Measures and Fit Indices for Model 1

MG-EFA	Number of Factors	Chi-Square	*df*	AIC	RMSEA	BIC
Model 1	1	1,553.773	130	1,293.773	0.103	2,153.114
	2	673.692	106	461.692	0.074	1,457.446
	3	195.633	84	27.633	0.043	1,148.432
	4	124.346	64	–3.654	0.032	1,230.822

Note. MG-EFA = multigroup exploratory factor analysis; AIC = Akaike's information criterion; RMSEA = root mean square error of approximation; BIC = Bayesian information criterion.

Table 2
Values of the Fit Measures and Fit Indices for Models 1, 2, 3, and 4

MG-EFA	Invariance Constraints, for all g	Chi-Square	*df*	AIC	RMSEA	BIC
Model 1	None	195.633	84	27.633	0.043	1,148.432
Model 2	$\mathbf{\Lambda}_g = \mathbf{\Lambda}$	245.732	114	17.732	0.044	968.015
Model 3	$\mathbf{\Lambda}_g = \mathbf{\Lambda}$ and $\boldsymbol{\nu}_g = \boldsymbol{\nu}$	265.548	124	17.548	0.044	910.992
Model 4	$\mathbf{\Lambda}_g = \mathbf{\Lambda}$, $\boldsymbol{\nu}_g = \boldsymbol{\nu}$, and $\mathbf{\Theta}_g = \mathbf{\Theta}$	288.741	137	14.742	0.044	834.295

Note. MG-EFA = multigroup exploratory factor analysis; AIC = Akaike's information criterion; RMSEA = root mean square error of approximation; BIC = Bayesian information criterion.

Nesting

In this section, it is shown that any MG-CFM is nested under the multigroup common factor model with minimal uniqueness constraints and the same number of common factors. The nesting of the MG-CFM under the multigroup common factor model with minimal constraints is important because it allows one to use the goodness of fit of the multigroup common factor model with minimal constraints as a baseline in comparisons with the more restrictive MG-CFM. In addition, this nesting means that power analyses can be carried out to gauge the effects of misspecification in the multigroup common factor model with minimal uniqueness constraints and the MG-CFM. In what follows, the nesting of MG-CFMs under the multigroup common factor model with minimal uniqueness constraints is for convenience also shown by means of the multigroup Model 4 (i.e., the multigroup model with strict factorial invariance conditions).

Let $\mathbf{M}_v$ be the invertible orthogonal varimax rotation matrix (Kaiser, 1958) for which $\mathbf{M}_v\mathbf{M}_v^t = \mathbf{I}$ and $\mathbf{M}_v^t = \mathbf{M}_v^{-1}$. Then, applying an orthogonal varimax rotation to Model 4 yields

$$\boldsymbol{\mu}_g = \boldsymbol{\nu} + \mathbf{\Lambda}\boldsymbol{\alpha}_g = \boldsymbol{\nu} + \mathbf{\Lambda}_v\boldsymbol{\beta}_g, \quad \text{for } g = 1, \ldots, G, \tag{13}$$

and

$$\mathbf{\Sigma}_g = \mathbf{\Lambda}\mathbf{\Psi}_g\mathbf{\Lambda}^t + \mathbf{\Theta} = \mathbf{\Lambda}_v\mathbf{\Phi}_g\mathbf{\Lambda}_v^t + \mathbf{\Theta}, \quad \text{for } g = 1, \ldots, G, \tag{14}$$

where $\mathbf{\Lambda}_v = \mathbf{\Lambda}\mathbf{M}_v$, $\boldsymbol{\beta}_g = \mathbf{M}_v^t\boldsymbol{\alpha}_g$, and $\mathbf{\Phi}_g = \mathbf{M}_v^{-1}\mathbf{\Psi}_g\mathbf{M}_v$, for all g. From the constraint $\boldsymbol{\alpha}_1 = \mathbf{0}$, it follows that $\boldsymbol{\beta}_1 = \mathbf{0}$ and

$$\begin{cases} \boldsymbol{\mu}_1 = \boldsymbol{\nu} \\ \boldsymbol{\mu}_g = \boldsymbol{\nu} + \mathbf{\Lambda}_v\boldsymbol{\beta}_g, \quad \text{for g} = 2, \ldots, \text{G}. \end{cases} \tag{15}$$

From $\mathbf{\Psi}_1 = \mathbf{I}$, it follows that $\mathbf{\Phi}_1 = \mathbf{M}_v^{-1}\mathbf{M}_v = \mathbf{I}$ and

$$\begin{cases} \mathbf{\Sigma}_1 = \mathbf{\Lambda}_v\mathbf{\Lambda}_v^t + \mathbf{\Theta} \\ \mathbf{\Sigma}_g = \mathbf{\Lambda}_v\mathbf{\Phi}_g\mathbf{\Lambda}_v^t + \mathbf{\Theta}, \quad \text{for } g = 2, \ldots, G. \end{cases} \tag{16}$$

Now let $\mathbf{M}_p$ be the oblique promax rotation matrix (Hendrickson & White, 1964). Then oblique promax rotation yields

$$\boldsymbol{\mu}_g = \boldsymbol{\nu} + \mathbf{\Lambda}_v\boldsymbol{\beta}_g = \boldsymbol{\nu} + \mathbf{\Lambda}_p\boldsymbol{\gamma}_g, \quad \text{for } g = 1, \ldots, G, \tag{17}$$

and

$$\mathbf{\Sigma}_g = \mathbf{\Lambda}_v\mathbf{\Phi}_g\mathbf{\Lambda}_v^t + \mathbf{\Theta} = \mathbf{\Lambda}_p\mathbf{\Omega}_g\mathbf{\Lambda}_p^t + \mathbf{\Theta}, \quad \text{for } g = 1, \ldots, G, \tag{18}$$

where $\mathbf{\Lambda}_p = \mathbf{\Lambda}_v\mathbf{M}_p$, $\boldsymbol{\gamma}_g = \mathbf{M}_p^{-1}\boldsymbol{\beta}_g$, and $\mathbf{\Omega}_g = \mathbf{M}_p^{-1}\mathbf{\Phi}_g(\mathbf{M}_p^t)^{-1}$, for all g. From the constraint $\boldsymbol{\beta}_1 = \mathbf{0}$, it follows that $\boldsymbol{\gamma}_1 = \mathbf{0}$ and

$$\begin{cases} \boldsymbol{\mu}_1 = \boldsymbol{\nu} \\ \boldsymbol{\mu}_g = \boldsymbol{\nu} + \mathbf{\Lambda}_p\boldsymbol{\gamma}_g, \quad \text{for } g = 2, \ldots, G. \end{cases} \tag{19}$$

The subpopulation covariance matrix remains

$$\mathbf{\Sigma}_g = \mathbf{\Lambda}_p\mathbf{\Omega}_g\mathbf{\Lambda}_p^t + \mathbf{\Theta}, \quad \text{for } g = 1, \ldots, G, \tag{20}$$

where for Group 1, $\mathbf{\Phi}_1 = \mathbf{I}$ and $\mathbf{\Omega}_1 = (\mathbf{M}_p^t\mathbf{M}_p)^{-1}$. Equations (19) and (20) include any prespecified confirmatory factor structure. Therefore, any MG-CFM is nested under the multigroup common factor model with minimal uniqueness constraints and the same number of common factors.

Power to Detect Uniform Bias

In this section, the purpose is to investigate the drop in power to detect uniform bias when a multigroup common factor model with minimal uniqueness constraints is used instead of a multigroup confirmatory factor model with a hypothesized true factor structure. This is important because in practice, the assumption of a true confirmatory factor structure may be hard to justify or can even be absent. Clearly, a serious drop in power to detect uniform bias in using the model with minimal constraints rather than the true confirmatory model would detract from the usefulness of the multigroup common factor model with minimal uniqueness constraints.

The fitted covariance matrices and the fitted mean vectors that follow from the confirmatory factor structure selected by Dolan (2000), in using the data set published by Jensen and Reynolds (1982), are used as the data for the power calculations. The fitted covariance matrices contain the model-predicted values of the observed covariances between the indicators, and the fitted mean vectors contain the model-predicted values of the observed indicator means.

Again, the selected confirmatory factor model includes strict factorial invariance over groups. Thus, $\boldsymbol{\Lambda}_g = \boldsymbol{\Lambda}$, $\boldsymbol{\nu}_g = \boldsymbol{\nu}$, and $\boldsymbol{\Theta}_g = \boldsymbol{\Theta}$, for both groups. Neither the factor mean vectors nor the covariance matrices are assumed to be invariant over groups. The confirmatory factor structure is selected as follows:

$$\boldsymbol{\Lambda}^t = \begin{bmatrix} \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} & 1 & \lambda_{5,1} & 0 & 0 & \lambda_{8,1} & \lambda_{9,1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{2,2} & 0 & 0 & \lambda_{5,2} & 0 & 0 & \lambda_{8,2} & \lambda_{9,2} & \lambda_{10,2} & 1 & \lambda_{12,2} & \lambda_{13,2} \\ \lambda_{1,3} & 0 & \lambda_{3,3} & 0 & 0 & \lambda_{6,3} & 1 & 0 & 0 & \lambda_{10,3} & 0 & \lambda_{12,3} & \lambda_{13,3} \end{bmatrix},$$

where the 0s and 1s are fixed values, and the λs are free to be estimated (Dolan, 2000). Fitting this MG-CFM to the data set published by Jensen and Reynolds (1982) yields a chi-square of 327.7 with 148 degrees of freedom, and the RMSEA is .033. Fitting either this MG-CFM or the more general two-group common factor model with minimal uniqueness constraints (Model 4) to the fitted covariance matrices and the fitted mean vectors will of course yield a perfect fit. This perfect fit will, however, disappear when the fitted mean vectors are manipulated to include uniform bias. Fitting both strict factorial invariance models (i.e., the two-group CFM and the two-group Model 4 with the same number of common factors) to the initial fitted covariance matrices and the initial fitted mean vectors as well as to the manipulated data provides a means to estimate the power to detect uniform bias.

Uniform bias is defined as one or more intercept differences between groups. In this power study, uniform bias is manipulated by adding a constant value of .25, .50, or .75 to one or two means in the group of Whites. These specific values are selected on the basis of calculated power values that can be classified as low ($\approx$ .25), medium ($\approx$ .55), or high ($\approx$ .85). Here, there is no need to explicitly explain these specific constant values in terms of an effect size. It suffices to show what happens to the power when the tests for uniform bias under the confirmatory model already have low, medium, or high power.

When two tests are selected to be uniformly biased, the constants are assigned the same value. When only one test is selected to be uniformly biased, the constant is added to the fitted mean of Test 1. When two tests are selected to be uniformly biased, the constants are added to the fitted means of Test 1 and Test 7. Furthermore, two levels of the total sample size are crossed with two levels of the sample size ratio between the two groups. The total sample size is selected to be either 600 or 1,200, and the sample size ratio is either equal to 1 (i.e., equal sample sizes for both groups) or unequal to 1 (i.e., unequal sample sizes for the two groups). In the unequal sample sizes conditions, the sample size of the Whites is chosen to be larger than the sample size of the Blacks. Next, both the two-group common factor Model 4 with strict factorial invariance and the overidentified confirmatory three-factor model, which was shown in the previous section to be a special case of Model 4, are fitted separately to test for uniform bias in all experimental conditions.

The power calculations are carried out by means of the method of Satorra and Saris (1985), which is based on the nonnull distribution of the (normal theory) log-likelihood ratio test. In the true model, one or more intercepts vary over groups. In the false model, which is a special case of the true model, strict factorial invariance over groups is assumed. The power $1 - \beta$ is the probability that the false model is rejected in favor of the true model. The power is approximated by $P\{\chi^2(r, \delta) > c_\alpha\}$, where r are the degrees of freedom, δ is the noncentrality parameter, and c_α is the critical value that depends on the nominal level of significance α (.05). The results of the power calculations are given in Table 3.

The results in Table 3 show that the approximated power increases with the total sample size and with the value of the constant added to the fitted means. In addition, the approximated power values are overall higher in the equal sample sizes conditions than in the unequal sample sizes conditions. Furthermore, in the case of two biased tests, the approximated power values are overall

Table 3

Approximated Power Values for the Detection of Uniform Bias Under Model 4

N_{Whites}	N_{Blacks}	Biased Tests	Constant	MG-CFM	Model 4
300	300	1	0.25	0.23	0.21
			0.50	0.68	0.64
			0.75	0.95	0.94
		2	0.25	0.27	0.23
			0.50	0.81	0.72
			0.75	0.99	0.98
400	200	1	0.25	0.22	0.21
			0.50	0.66	0.62
			0.75	0.94	0.92
		2	0.25	0.25	0.22
			0.50	0.77	0.70
			0.75	0.99	0.97
600	600	1	0.25	0.41	0.38
			0.50	0.93	0.91
			0.75	1.00	1.00
		2	0.25	0.50	0.42
			0.50	0.98	0.96
			0.75	1.00	1.00
900	300	1	0.25	0.35	0.33
			0.50	0.88	0.86
			0.75	1.00	0.99
		2	0.25	0.41	0.36
			0.50	0.95	0.92
			0.75	1.00	1.00

Note. MG-CFM = multigroup confirmatory factor model.

higher than in case of one biased test. Finally, the drop in approximated power to detect uniform bias in using the multigroup common factor model with minimal uniqueness constraints instead of the MG-CFM is not greater than .09 in each condition. The mean difference in approximated power between the two models over all conditions is .03.

Discussion

In this article, an alternative formulation is proposed for the multigroup common factor model with minimal uniqueness constraints (McArdle & Cattell, 1994). Although this alternative multigroup common factor analysis solution does not produce other values of the fit measures, the fit indices, and the degrees of freedom, it has the advantage of less technical difficulties in applications over earlier formulations of the model (McArdle & Cattell, 1994). The advantage is due to the constraint that the determinant $|\mathbf{\Lambda}^t\mathbf{\Lambda}|$ reaches a maximum. This constraint is related to the standard ML constraint in single-group common factor analysis (Browne, 1969). McArdle and Cattell (1994) mention the ML constraint in passing but devote most of their attention to other identifying constraints. Presumably, this was because they wanted to provide a comprehensive account of identifying constraints in the multigroup common factor model and perhaps also because the standard constraint is not easy to specify in traditional programs such as LISREL (Jöreskog & Sörbom, 2002). In the Mx computer program (Neale et al., 1999), however, both identification constraints do not pose any computational difficulties.

The multigroup common factor model with minimal uniqueness constraints has very useful applications in the study of group differences in cognitive abilities or personality. The model provides a means to investigate various degrees of measurement invariance over groups, including the invariance of all factor loadings. Consequently, the model can also be used to test for factorial congruence. By adopting an explicit multigroup common factor modeling approach, factorial congruence can be evaluated, both as an omnibus hypothesis and with respect to individual common factors by means of a likelihood ratio test. In addition, the multigroup common factor model with minimal constraints provides a useful baseline by which the goodness of fit of any MG-CFM may be evaluated. For instance, it may be of interest to compare the goodness of fit of the metric invariance multigroup common factor model with minimal constraints with that of a metric MG-CFM.

In this article, the focus has mainly been on the usefulness of the multigroup common factor model with minimal uniqueness constraints for the detection of uniform bias. Less attention has been given to the other tests necessary to establish strict factorial invariance over groups. The investigation of the usefulness of the multigroup common factor model with minimal uniqueness constraints for these other tests will probably yield similar results and is left to possible future studies. Here, the usefulness of the model for the detection of uniform bias has been investigated by means of power calculations. When the application of the model would automatically mean that the power to detect uniform bias is significantly low compared to the power associated with the application of the more restrictive true MG-CFM, then the use of the model for this purpose could be questioned. The results of the power calculations in the present study, however, show that the drop in power is small. Therefore, when a strong confirmatory factor structure is absent or unknown, the use of the multigroup common factor model with minimal uniqueness constraints can be recommended in a standard procedure to test for factorial invariance over groups.

Appendix

Mx code for the strict factorial invariance Model 4:

```
#ngroups 3

Whites
da no=1868 ni=13 ng=3
cm file=whitec.mx
me file=whitem.mx
begin matrices
L fu 13 3 fr    ! factor pattern matrix
P st 3 3 fi    ! common factor covariance matrix
t di 13 13 fr    ! standard deviations of residuals
a fu 3 1 fi    ! vector of factor means
n fu 13 1 fr    ! vector of intercepts
end matrices
st 1.5 L 1 1 to L 13 3
```

```
st 3.5 t 1 1 to t 13 13
st 10 n 1 1 to n 13 1
co L*P*L' + t*t;
me n';
end group

necessary constraint
con ni=13
begin matrices;
L fu 13 3 = L1
I id 3 3
end matrices;
constraint I.(L'*L) = L'*L;
end group

Blacks
da no = 305 ni = 13
cm file = blackc.mx
me file = blackm.mx
begin matrices
L fu 13 3 = L1
P sy 3 3 fr
t di 13 13 = t1
a fu 3 1 fr
n fu 13 1 = n1
end matrices
st 2.0 p 1 1 p 2 2 p 3 3
st 1.0 p 2 1 p 3 1 p 3 2
st 0.0 a 1 1 to a 3 1
co L*P*L' + t*t;
me (n + L*a)';
option iterations=10000
end group
```

References

Browne, M. W. (1969). Fitting the factor analysis model. *Psychometrika, 34,* 375-394.

Chan, W., Ho, R. M., Leung, K., Chan, D. K., & Yung, Y. (1999). An alternative method for evaluating congruence coefficients with Procrustes rotation: A bootstrap procedure. *Psychological Methods, 4,* 378-402.

Dolan, C. V. (2000). Investigating Spearman's hypothesis by means of multi-group confirmatory factor analysis. *Multivariate Behavioral Research, 35,* 21-50.

Dolan, C. V., & Molenaar, P. C. M. (1994). Testing specific hypotheses concerning latent group differences in multi-group covariance structure analysis with structured means. *Multivariate Behavioral Research, 29,* 203-222.

Hendrickson, A. E., & White, P. O. (1964). Promax: A quick method for rotation to oblique simple structure. *British Journal of Statistical Psychology, 17,* 65-70.

Horn, J. L., & McArdle, J. J. (1992). A practical and theoretical guide to measurement invariance in aging research. *Experimental Aging Research, 18,* 117-144.

Jensen, A. R., & Reynolds, C. R. (1982). Race, social class and ability patterns on the WISC-R. *Personality and Individual Differences, 3,* 423-438.

Jöreskog, K. G. (1971). Simultaneous factor analysis in several populations. *Psychometrika, 36,* 409-426.

Jöreskog, K. G., & Sörbom, D. (2002). *LISREL 8: Structural equation modeling with the SIMPLIS command language*. Chicago: Scientific Software International.

Kaiser, H. F. (1958). The varimax criterion for analytic rotation in factor analysis. *Psychometrika, 23,* 187-200.

Lawley, D. N., & Maxwell, A. E. (1971). *Factor analysis as a statistical method.* London: Butterworth & Co.

McArdle, J. J., & Cattell, R. B. (1994). Structural equation models of factorial invariance in parallel proportional profiles and oblique confactor problems. *Multivariate Behavioral Research, 29,* 63-113.

Mellenbergh, G. J. (1994a). Generalized linear item response theory. *Psychological Bulletin, 115,* 300-307.

Mellenbergh, G. J. (1994b). A unidimensional latent trait model for continuous item responses. *Multivariate Behavioral Research, 29,* 223-236.

Meredith, W. (1993). Measurement invariance, factor analysis, and factorial invariance. *Psychometrika, 58,* 525-543.

Millsap, R. E. (2001). When trivial constraints are not trivial: The choice of uniqueness constraints in confirmatory factor analysis. *Structural Equation Modeling, 8,* 1-17.

Neale, M. C., Boker, S. M., Xie, G., & Maes, H. H. (1999). *Mx: Statistical modeling* (5th ed.). Richmond, VA: Department of Psychiatry.

Satorra, A., & Saris, W. E. (1985). Power of the likelihood ratio test in covariance structure analysis. *Psychometrika, 50,* 83-90.

Schafer, J. L., & Graham, J. W. (2002). Missing data: Our view of the state of the art. *Psychological Methods, 7,* 147-177.

Shealy, R. T., & Stout, W. F. (1993). An item response theory model for test bias and differential item functioning. In P. W. Holland & H. Wainer (Eds.), *Differential item functioning* (pp. 197-239). Hillsdale, NJ: Lawrence Erlbaum.

Sörbom, D. (1974). A general method for studying differences in factor means and factor structure between groups. *British Journal of Mathematical and Statistical Psychology, 27,* 229-239.

Widaman, K. F., & Reise, S. P. (1997). Exploring the measurement invariance of psychological instruments: Applications in the substance use domain. In K. J. Bryant, M. Windle, & S. G. West (Eds.), *The science of prevention: Methodological advances from alcohol and substance abuse research* (pp. 281-324). Washington, DC: American Psychological Association.

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