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A Note on the Relationship Between the Number of Indicators and Their Reliability in Detecting Regression Coefficients in Latent Regression Analysis

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We consider the question of how variation in the number and reliability of indicators affects the power to reject the hypothesis that the regression coefficients are zero in latent linear regression analysis. We show that power remains constant as long as the coefficient of determination remains unchanged. Any increase in the number of indicators always results in an increase in the coefficient of determination and so in the power. We note that the coefficient of determination plays a similar role in determining the error variance of predicted factor scores.

Imagine that one wishes to regress the latent variable η on the latent variable ξ. In deciding on the number of indicators to use to measure η and ξ, one may want to maximize the power to reject the omnibus null hypothesis that the regression coefficients are zero. Given this aim, is one better off with many unreliable indicators, or with few reliable indicators? The aim of this article is to consider this issue in multivariate latent regression analysis. We define reliability as the ratio of the variance of an indicator explained by the common factor to its total (modeled) variance. We limit our focus to the criterion of statistical power, that is, the probability that we reject the *omnibus hypothesis* that the regression coefficients are zero, given that in truth they are nonzero. We first present the general model along with our assumptions. Subsequently we introduce the loglikelihood ratio function. To

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address this issue, we consider various representations of this function. Throughout we make use of elementary matrix algebra (e.g., Bollen, 1989; Lawley & Maxwell, 1971) and some results concerning matrix inverses and determinants (e.g., Schott, 1997).

MODEL

Consider the following regression model (in LISREL notation; e.g., Jöreskog & Sörbom, 1993):

$$\eta_i = \Gamma \xi_i + \zeta_i \tag{1}$$

where η_i and ξ_i are the n_e-dimensional dependent and n_k-dimensional independent latent (random) variables, respectively, Γ is the $n_e \times n_k$ matrix of regression coefficient, and ζ_i is a n_e-dimensional vector of random residuals. The subscript i denotes case ($i = 1 \ldots N$). We measure the latent variables by means of n_y and n_x indicators of η and ξ, respectively. The measurement models are common factor models:

$$\mathbf{y}_i = \Lambda_y \eta_i + \varepsilon_i \tag{2}$$

$$\mathbf{x}_i = \Lambda_x \xi_i + \delta_i \tag{3}$$

where the vector $\mathbf{y}_i$ contains the observed scores of subject i, the $n_y \times n_e$ matrix L_y contains the factor loadings, and ε_i is the n_y dimensional vector of uncorrelated residuals. We assume that all variables have zero mean. In matrix notation we have the following covariance matrix:

$$\Sigma_0 = \begin{bmatrix} \Lambda_x \Phi \Lambda_x{}' + \Theta_x & \Lambda_x\ \Phi\Gamma'\ \Lambda_y{}' \\ \Lambda_y \Gamma \Phi \Lambda_x{}' & \Lambda_y \Sigma_\eta \Lambda_y{}' + \Theta_y \end{bmatrix} = \begin{bmatrix} \Sigma_x & \Sigma_{yx}{}' \\ \Sigma_{yx} & \Sigma_y \end{bmatrix} \tag{4}$$

where the Φ is the $n_k \times\ n_k$ covariance matrix of ξ. The matrix Σ_η equals $[\Gamma\Phi\Gamma' + \Psi]$, and Ψ is the $n_e \times n_e$ covariance matrix of ζ. Θ_x and Θ_y are the $n_x \times n_x$ and $n_y \times n_y$ covariance matrices of δ_i and ε_i, respectively. We assume that Λ_x, Λ_y, and Γ have full column rank, that the Ψ are Φ positive definite, and that Θ_x are Θ_y diagonal and positive. Furthermore, we assume that each indicator loads on one common factor. This assumption implies that Λ_x and Λ_y display simple structure and that the matrices $\Lambda_x{}'\ \Theta_x{}^{-1}\Lambda_x$ and $\Lambda_y{}'\ \Theta_y{}^{-1}\Lambda_y$ are diagonal and positive. The focus of the analysis is the power to show that Γ differs from zero. Fixing this parameter matrix to zero ($\Gamma = 0$), we have the following expected, or model, covariance matrix:

$$\Sigma_1 = \begin{bmatrix} \Sigma_x & \mathrm{O}' \\ \mathrm{O} & \Sigma_y \end{bmatrix} \tag{5}$$

where O is an $n_y \times n_x$ zero matrix. In Equation 5, Σ_x simply equals $\Lambda_x\Phi\Lambda_x' + \Theta_x$ (as in Equation 4). Σ_y equals $\Lambda_y[\Gamma\Phi\Gamma' + \Psi]\Lambda_y' + \Theta_y$ but is modeled as $\Lambda_y\Sigma_\eta\Lambda_y' + \Theta_y$. In the following we exploit the fact that the unconstrained covariance matrix Σ_η actually equals $[\Gamma\Phi\Gamma' + \Psi]$. With respect to this model, assume that the addition of an indicator results in a change in Λ_y and/or Λ_x and in Θ_y and/or Θ_x, but the other parameters remain unaffected. The relation between the numbers of indicators and the reliability of the indicators may be investigated by considering the question of when the power to reject the hypothesis $\Gamma = \mathbf{0}$ remains constant given variation in number and the reliability of the indicators.

WHEN IS POWER CONSTANT?

Assuming the data are multinormally distributed, we focus on the noncentral chi-square distribution as the nonnull distribution of the test statistic. We assume that sample size N and Type 1 error probability, α, are known (i.e., assigned relevant values). As explained by Saris and Satorra (1993) to calculate power in the situation, we require the noncentrality parameter (ncp) λ of the noncentral chi-square distribution. The number of degrees of freedom and λ determine the nonnull distribution. In this case, the number of degrees of freedom equals the number of elements in Γ ($n_e \times n_k$). Once the null (central chi-square) and nonnull (noncentral chi-square) distributions are known, we can calculate the power. As explained in Saris and Satorra, an approximation of the parameter λ may be obtained by fitting the incorrect model (Equation 5) to the population covariance matrix (Equation 4).

Given this model, the normal theory loglikelihood ratio function, which yields the approximation of λ, is (Bollen, 1989):

$$\lambda \approx \{n^* [\log|\Sigma_1| + \text{trace}(\Sigma_1^{-1}\Sigma_0) - \log|\Sigma_0| - (n_y + n_x)]\} \tag{6}$$

where $n = (N - 1)$. It can be verified readily that trace $(\Sigma_1^{-1}\Sigma_0)$ equals $(n_y + n_x)$. Furthermore, as $|\Sigma_1| = |\Sigma_x||\Sigma_y|$, we have

$$\lambda \approx n^* \log(|\Sigma_x||\Sigma_y| / |\Sigma_0|) \tag{7}$$

We may further simplify the expression given the following results (e.g., Schott, 1997, p. 250):

$$|\Sigma_0| = |\Sigma_x||\Sigma_y - \Sigma_{yx}'\Sigma_x^{-1}\Sigma_{yx}| \tag{8a}$$

$$|\Sigma_0| = |\Sigma_y||\Sigma_x - \Sigma_{yx}\Sigma_y^{-1}\Sigma_{yx}'| \tag{8b}$$

Applying Equation 8b and simplifying, we have:

$$\lambda \approx n^* \log(|\Sigma_x| / |\Sigma_x - \Sigma_{yx}\Sigma_y^{-1}\Sigma_{yx}'|) \tag{9}$$

We now make use of the following results. The inverse of $\Sigma_x = \Lambda_x\Phi_x\Lambda_x' + \Theta x$ may be expressed as follows:

$$[\Lambda_x\Phi_x\Lambda_x' + \Theta_x]^{-1} = \Theta_x^{-1} - \Theta_x^{-1}\Lambda_x[\Phi_x^{-1} + \Lambda_x'\,\Theta_x^{-1}\Lambda_x]^{-1}\Lambda_x'\,\Theta_x^{-1} \tag{10}$$

(e.g., see Schott, 1997, p. 9), and

$$|\Sigma_x| = |\Theta_x||I + \Phi_x^{1/2\prime}\,\Lambda_x'\Theta_x^{-1}\Lambda_x\Phi^{1/2}| \tag{11}$$

(e.g., Lawley & Maxwell, 1971, Exercise 4.1), where I is the identity matrix, and $\Phi^{1/2\prime}\,\Phi^{1/2} = \Phi$. For instance, $\Phi^{1/2}$ may be the Cholesky decomposition, in which case $\Phi^{1/2}$ is an upper triangular matrix. We apply Equation 10 and rearrange

$$|\Sigma_x - \Sigma_{yx}\,\Sigma_y^{-1}\Sigma_{yx}'| = |\Lambda_x\Phi\Lambda_x' + \Theta_x - \Lambda_x\Phi\Gamma'\Lambda_y'\,(\Lambda_y\Sigma_\eta\,\Lambda_y' + \Theta_y)^{-1}\,\Lambda_y\Gamma\Phi\Lambda_x'| =$$

$$|\,\Lambda_x\Phi\Lambda_x' - \Lambda_x\Phi\Gamma'\Lambda_y[\Theta_y^{-1} - \Theta_y^{-1}\Lambda_y[\Sigma_\eta^{-1} + \Lambda_y'\,\Theta_y^{-1}\Lambda_y]^{-1}\Lambda_y'\,\Theta_y^{-1}]\,\Lambda_y\Gamma\Phi\Lambda_x' + \Theta_x|.$$

To simplify presentation, let Δ_x and Δ_y equal the diagonal matrices $\Lambda_x'\,\Theta_x^{-1}\Lambda_x$ and $\Lambda_y'\,\Theta_y^{-1}\Lambda_y$, respectively:

$$|\Sigma_x|/|\Sigma_x - \Sigma_{yx}\,\Sigma_y^{-1}\Sigma_{yx}'| = |\Lambda_x\Phi\Lambda_x' + \Theta_x|/|\Lambda_x\,\Xi\,\Lambda_x' + \Theta_x|$$

where $\Xi = \{\Phi - \Phi\Gamma'\,[\Delta_y - \Delta_y(\Sigma_\eta^{-1} + \Delta_y)^{-1}\Delta_y]\,\Gamma\Phi\}$

Given Equation 11, we have

$$|\Sigma_x|/|\Sigma_x - \Sigma_{yx}\,\Sigma_y^{-1}\Sigma_{yx}'| = |\Theta_x||I + \Phi^{1/2\prime}\Delta_x\,\Phi^{1/2}|/|\Theta_x||I + \Xi^{1/2\prime}\Delta_x\,\Xi^{1/2}|$$

or

$$|\Sigma_x|/|\Sigma_x - \Sigma_{yx}\,\Sigma_y^{-1}\Sigma_{yx}'| = |I + \Phi^{1/2\prime}\Delta_x\,\Phi^{1/2}|/|I + \Xi^{1/2\prime}\Delta_x\,\Xi^{1/2}|$$

We thus arrive at a number of equivalent expressions for Equation 6:

$$\lambda \approx n^*\,\log(|\Lambda_x\Phi\Lambda_x' + \Theta_x|/|\Lambda_x\,\Xi\,\Lambda_x' + \Theta_x|) \tag{12}$$

$$\lambda \approx n^*\,\log(|I + \Phi^{1/2\prime}\Delta_x\,\Phi^{1/2}|/|I + \Xi^{1/2\prime}\Delta_x\,\Xi^{1/2}|) \tag{13}$$

Applying Equation 11 to Equation 13, we have

$$\lambda \approx n^*\,\log\,(|I + \Delta_x^{1/2}\Phi\,\Delta_x^{1/2}|/|I + \Delta_x^{1/2}\Xi\,\Delta_x^{1/2}|) \tag{14}$$

Finally, we may write (Harville, 1997, p. 416)

$$\lambda \approx n^*\,\log\,(|I + \Delta_x\Phi|/|I + \Delta_x\Xi|) \tag{15}$$

As noted, the addition of an indicator of a component of ξ (η) results in changes in Λ_x and Θ_x (Λ_y and Θ_y). Equations 13 to 15 reveal immediately that variations in the number of indicators and/or their reliabilities will not affect the ncp λ, as long as Δ_x and Δ_y (i.e., $\Lambda_x'\,\Theta_x^{-1}\Lambda_x$ and $\Lambda_y'\,\Theta_y^{-1}\Lambda_y$) remain constant. We note that any increase in Δ_x and/or Δ_y will result in an increase in power. In the case of Δ_x, this follows from the fact in Equation 15 that $\Phi > \Xi$, in the sense that $\Phi - \Xi$ is positive (semi-)definite. This can be demonstrated as follows. We consider $\Phi - \Xi = \{\Phi\Gamma'[\Delta_y - \Delta_y(\Sigma_\eta^{-1} + \Delta_y)^{-1}\Delta_y]\Gamma\Phi\}$. We first note that $[\Delta_y - \Delta_y(\Sigma_\eta^{-1} + \Delta_y)^{-1}\Delta_y]$ equals $\Delta_y[\Delta_y^{-1} - (\Sigma_\eta^{-1} + \Delta_y)^{-1}]\Delta_y$. As matrix Δ_y is positive definite, we consider $[\Delta_y^{-1} - (\Sigma_\eta^{-1} + \Delta_y)^{-1}]$. This matrix is positive definite by present assumptions: We postmultiply this with the positive definite matrix $(\Sigma_\eta^{-1} + \Delta_y)$, so that we end up with $\Delta_y^{-1}\Sigma_\eta^{-1}$. The matrix $\Delta_y^{-1}\Sigma_\eta^{-1}$ is positive definite by present assumptions. This demonstrates that $[\Delta_y - \Delta_y(\Sigma_\eta^{-1} + \Delta_y)^{-1}\Delta_y]$ is positive definite.[1] As Φ is positive definite and Γ has full column rank, the matrix $(\Phi - \Xi)$ is positive semidefinite in the event that $n_k > n_e$, and positive definite when $n_k \leq n_e$. So any increase in Δ_x will result in an increase in the ncp λ, and thus in power.

By using Equation 8a rather than 8b, and going through the same steps as previously, one can arrive at an alternative expression for Equation 6. For instance, the function may be written as follows:

$$\lambda = n^* \log (|I + \Delta_y^{1/2}\Sigma_\eta \Delta_y^{1/2}|/|I + \Delta_y^{1/2}\Omega \Delta_y^{1/2}|)$$

where $\Omega = \{\Gamma\Phi\Gamma' + \Psi - \Gamma\Phi[\Delta_x - \Delta_x(\Phi^{-1} + \Delta_x)^{-1}\Delta_x]\Phi\Gamma'\}$. Using the same line of reasoning one may demonstrate that $\Sigma_\eta - \Omega$ is positive semidefinite, and so any increase in Δ_y will result in an increase in the ncp λ, and thus in power.

To illustrate, we focus on the predictors, and suppose that $n_k = 1$. Note that the scalar Δ_x $(\Lambda_x' \Theta_x^{-1}\Lambda_x)$ equals

$$\Delta_x = \sum_{k=1}^{n_x}[\lambda_{x\{k\}}{}^2 / \sigma^2{}_{x\{k\}}]$$

where $\lambda_{x(k)}$ and $\sigma^2{}_{x(k)}$ are the kth component of Λ_x and the kth diagonal element in Θ_x. The addition of indicator $n_x + 1$ will necessarily increase Δ_x by $[\lambda_{x(nx+1)}{}^2/\sigma^2{}_{x(nx+1)}]$. Clearly the number of indicators itself is not informative concerning power. For instance, $\Lambda_x = [1,1,1]'$ and $\Theta_x = \text{diag}[.5, .5, .5]$ confer the same power as $\Lambda_x = [1, 1, 1, 1, 1, 1]'$ and $\Theta_x = \text{diag}[1, 1, 1, 1, 1, 1]$, because in both cases Δ_x equals 6.

Finally consider the situation in which Δ_x equals Δ_y. Here a constant increase in Δ_x or in Δ_y will result in an identical increase in power to reject $\Gamma = \mathbf{0}$. In other words, given the hypothesis of $\Gamma = \mathbf{0}$ and assuming Δ_x equals Δ_y, there is no advantage in adding an indicator to ξ over adding an indicator to η.

DISCUSSION

Given these assumptions, we have found that as long as the diagonal matrices Δ_x and Δ_y remain constant, the power to reject the hypothesis that $\Gamma = \mathbf{0}$ remains con-

[1]A more rigorous proof of this is due to our friend Raoul Grasman. This proof is available on request.

stant. This finding is plausible as it implies that the coefficient of determination remains constant. The coefficient of determination may be viewed as the multivariate version of the reliability (Bollen, 1989, p. 289). It is calculated as follows (see Equation 11):

$$1 - |\Theta|/|\Sigma| = 1 - (|\Theta|/|\Theta||\mathbf{I} + \Phi^{-1/2\prime}\Lambda'\Theta^{-1}\Lambda\Phi^{-1/2}|) = 1 - |\mathbf{I} + \Phi^{-1/2\prime}\Lambda'\Theta^{-1}\Lambda\Phi^{-1/2}|^{-1}$$

In the case of a single factor model, where the common factor is standardized, we have $1 - |\Theta|/|\Sigma| = 1 - [1 + \Lambda'\Theta^{-1}\Lambda]^{-1}$ (see Equation 11). Note that $\Lambda'\Theta^{-1}\Lambda$ may be viewed here as the signal-to-noise ratio, where the signal is variance due to the common factor, and the noise is variance due to the residuals.

Interestingly, the matrix $\Lambda'\Theta^{-1}\Lambda$ also features in the context of factor score prediction. Specifically, in the case of uncorrelated common factors, the matrix $(I + \Lambda'\Theta^{-1}\Lambda)^{-1}$ is the variance of the errors in the prediction the common factor scores from observed scores on the indicators (see Lawley & Maxwell, 1971, p. 107). So we have the same result with respect to the precision of the predicted factor scores as with the power to reject $\Gamma = \mathbf{0}$: The error variance of the predicted factor scores remains constant as long as the coefficient of determination remains constant. The result is limited to the error covariance matrix of factor scores calculated according to the regression method (for other methods, see Lawley & Maxwell, 1971; Saris, de Pijper, & Mulder, 1978).

These results concern a single issue. They are predicated on, and limited to, the particular regression model, which we presented previously, and the omnibus null hypothesis of $\Gamma = \mathbf{0}$. This omnibus hypothesis gives rise to a block diagonal covariance matrix Σ_1 (Equation 5), which greatly facilitates the derivation of the loglikelihood ratio functions and, thus, of the ncp λ. These results will hold for other models in which the nonnull hypothesis gives rise to a block diagonal Σ_1. However, in more complicated situations (e.g., the test that a given component of Γ is zero), it may be difficult to obtain simple results. In such situations, one can always carry out detailed power calculations as outlined in Saris and Satorra (1993). Regardless of the generality of the results, such calculations are potentially a rich source of information. Even in the event that a given model is never actually fitted to data, power calculations provide information concerning the possible practical limitations of the model and, by extension, of the attendant theory.

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